

Real algebraic geometry, moment problems and multivariate tight wavelet frames

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Abstract

In this paper we employ recent results from real algebraic geometry and theory of moment problems to make the first step towards resolving the question of existence of multivariate tight wavelet frames whose generators have at least one vanishing moment. The so-called Unitary Extension Principle from [34] and the results in [32] allow us to reformulate the question of existence of tight wavelet frame in terms of the existence of the sum of squares decomposition of a single trigonometric polynomial with real coefficients. Our main result confirms the existence of such decompositions in the two-dimensional case. We also give sufficient conditions for existence of tight wavelet frames in the dimension $d \geq 3$ and illustrate our results with several examples.

Keywords: multivariate wavelet frames, real algebraic geometry, moment problems

1 Introduction

There are several fundamental results by two groups of authors I. Daubechies, B. Han, A. Ron, Z. Shen [17] and C. Chui, W. He, J. Stöckler [9, 10] that build up the theory of tight wavelet frames and also provide their characterizations. Those characterizations, on the one hand, allow us to establish the connection between frame constructions and a difficult algebraic problem of existence of sums of squares representations (sos) of non-negative trigonometric polynomials. On the other hand, these characterizations also offer methods, however unsatisfactory from the practical point of view, for construction of tight wavelet frames.

The existence and practical methods for construction of tight frames, together with good estimates on the number of frame generators, are still open problems. On the one hand, one could get discouraged by a general result by Scheiderer in [37], which implies that not all non-negative trigonometric polynomials in the dimension $d \geq 3$ possess sos representations. On the other hand, we are dealing mostly with the case $d = 2$ and non-negative trigonometric polynomials with special properties. This motivates us to pursue the issue of existence of sos representations further.

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It has been observed in [15] that redundancy of wavelet frames has advantages for applications in signal denoising - if the data is redundant, then losing some data during transmission does not necessarily effect the reconstruction of the original signal. Shen et al. [19] use the tight wavelet frame decomposition to recover a clear image from a single motion-blurred image. In [1] the authors show how to use multiresolution wavelet filters P and Q_j to construct irreducible representations for the Cuntz algebra and, conversely, how to recover wavelet filters from these representations. Wavelet and frame decompositions for subdivision surfaces are one of the basic tools, e.g., for progressive compression of 3-d meshes or interactive surface viewing [5, 28, 33]. Adaptive numerical methods based on wavelet frame discretizations have yielded very promising results [12, 13] when applied to a large class of operator equations, in particular, PDE and integral equations.

In this paper we employ recent results from real algebraic geometry and theory of moment problems to make the first step towards resolving the question of existence of multivariate tight wavelet frames with generators having at least one vanishing moment. In other words the foundation for our results is the so-called Unitary Extension Principle from [34], a special case of the above mentioned characterizations in [9, 10, 17]. In section 3, we give several equivalent formulations of UEP that allow us to reformulate the problem of construction of tight wavelet frames as a problem of semi-definite programming. This establishes a connection between constructions of tight wavelet frames and moment problems, see [24, 29, 30] for details. In section 4.1, using results of [36], we show that the existence of such tight wavelet frames in the two-dimensional case is always guaranteed. We also present an elegant method that sometimes simplifies the actual frame construction and illustrate this method on the example of the so-called butterfly scheme from [20]. In section 4.2, we give a sufficient condition for the existence of tight wavelet frames in dimension $d \geq 3$ and illustrate our results with several examples of three-dimensional subdivision. We also show that not all 3-dimensional trigonometric polynomials under consideration can be written as sums of squares of trigonometric polynomials with real coefficients.

We list some existing constructions of compactly supported MRA wavelet tight frames of $L_2(\mathbb{R}^d)$ [8, 11, 17, 23, 32, 34, 38] that employ the Unitary Extension Principle. For any dimension and in the case of a general expansive dilation matrix, the existence of tight wavelet frames is always ensured by [3, 4], if the coefficients of the associated refinement equation are real and nonnegative. There are only few compactly supported multi-wavelet tight frames in the literature, see [3, 5, 22]. In [3] the authors present a general method for constructing multi-wavelet tight frames in the case when the matrix coefficients of P all have only non-negative entries. The results of [5] are based on the time-domain techniques developed in [9, 10] and boil down to symmetric factorizations of local positive semi-definite real matrices. The method in [5] for matrix-valued P is applicable only in the special case, when the refinement coefficients satisfy identity [5, (9)]. Another construction approach is given in [22] and is an adaptation of the method based on singular-value decompositions first presented in [31].

2 Background and Notation

Let $d \in \mathbb{N}$, let T denote the d -dimensional anisotropic real (algebraic) torus, and let $\mathbb{R}[T]$ denote the (real) affine coordinate ring of T

$$\mathbb{R}[T] = \mathbb{R}[x_j, y_j : j = 1, \dots, d] / (x_j^2 + y_j^2 - 1 : j = 1, \dots, d).$$

Rather than working with this description, we will mostly employ the complexification of T , together with its affine coordinate ring $\mathbb{C}[T] = \mathbb{R}[T] \otimes_{\mathbb{R}} \mathbb{C}$. This coordinate ring comes with a natural \mathbb{C}/\mathbb{R} -involution $*$ on $\mathbb{C}[T]$, induced by complex conjugation. Namely,

$$\mathbb{C}[T] = \mathbb{C}[z_1^{\pm 1}, \dots, z_d^{\pm 1}]$$

is the ring of complex Laurent polynomials, and $*$ sends z_j to z_j^{-1} and is complex conjugation on coefficients. The real coordinate ring $\mathbb{R}[T]$ consists of the $*$ -invariant polynomials in $\mathbb{C}[T]$.

The group of \mathbb{C} -points of T is $T(\mathbb{C}) = (\mathbb{C}^*)^d = \mathbb{C}^* \times \dots \times \mathbb{C}^*$. In this paper we often denote the group of \mathbb{R} -points of T by \mathbb{T}^d . Therefore,

$$\mathbb{T}^d = T(\mathbb{R}) = \{(z_1, \dots, z_d) \in (\mathbb{C}^*)^d : |z_1| = \dots = |z_d| = 1\}$$

is the direct product of d copies of the circle group S^1 . The neutral element of this group we denote by $\mathbf{1} = (1, \dots, 1)$.

Via the exponential map \exp , the coordinate ring $\mathbb{C}[T] = \mathbb{C}[z_1^{\pm 1}, \dots, z_d^{\pm 1}]$ of T is identified with the algebra of (complex) trigonometric polynomials. Namely, \exp identifies z_j with $e^{-i\omega_j}$, $j = 1, \dots, d$. In the same way, the real coordinate ring $\mathbb{R}[T]$ is identified with the ring of real trigonometric polynomials, i.e. polynomials with real coefficients in $\cos(\omega_j)$ and $\sin(\omega_j)$, $j = 1, \dots, d$.

Let $M \in \mathbb{Z}^{d \times d}$ be a general expansive matrix, i.e. $\rho(M^{-1}) < 1$, or equivalently, all eigenvalues of M are strictly larger than 1 in modulus. Let $m := |\det M|$. The finite abelian group $G := 2\pi M^{-T} \mathbb{Z}^d / 2\pi \mathbb{Z}^d$ has order $|G| = m$, its character group is $G' = \mathbb{Z}^d / M \mathbb{Z}^d$.

A wavelet tight frame is a structured system of functions that has some special group structure and is defined by the actions of translates and dilates on a finite set of functions $\psi_j \in L_2(\mathbb{R}^d)$, $1 \leq j \leq N$. More precisely, we define translation operators T_α on $L_2(\mathbb{R}^d)$ by $T_\alpha f = f(\cdot - \alpha)$, $\alpha \in \mathbb{Z}^d$, and dilation (homothety) U_M by $U_M f = m^{1/2} f(M \cdot)$. Note that these operators are isometries on $L_2(\mathbb{R}^d)$.

Definition 2.1. Let $\{\psi_j : 1 \leq j \leq N\} \subseteq L_2(\mathbb{R}^d)$. The family

$$\Psi = \{U_M^\ell T_\alpha \psi_j : 1 \leq j \leq N, \ell \in \mathbb{Z}, \alpha \in \mathbb{Z}^d\}$$

is a wavelet tight frame of $L_2(\mathbb{R}^d)$, if

$$\|f\|_{L_2}^2 = \sum_{\substack{1 \leq j \leq N, \ell \in \mathbb{Z}, \\ \alpha \in \mathbb{Z}^d}} |\langle f, U_M^\ell T_\alpha \psi_j \rangle|^2 \quad \text{for all } f \in L_2(\mathbb{R}^d). \quad (1)$$

The foundation for the construction of multiresolution wavelet basis or wavelet tight frame is a compactly supported real-valued function $\phi \in L_2(\mathbb{R}^d)$ with the following properties.

- (i) ϕ is refinable, i.e. there exists a finitely supported sequence $p = (p(\alpha))_{\alpha \in \mathbb{Z}^d}$, $p(\alpha) \in \mathbb{R}$, such that ϕ satisfies

$$\phi(x) = m^{-1/2} \sum_{\alpha \in \mathbb{Z}^d} p(\alpha) U_M T_\alpha \phi(x). \quad x \in \mathbb{R}^d. \quad (2)$$

Taking the Fourier-Transform

$$\widehat{\phi}(\omega) = \int_{\mathbb{R}^d} \phi(x) e^{-i\omega \cdot x} dx$$

of both sides of (2) leads to its equivalent form

$$\widehat{\phi}(M^T \omega) = P(e^{-i\omega}) \widehat{\phi}(\omega), \quad e^{-i\omega} = (e^{-i\omega_1}, \dots, e^{-i\omega_d}), \quad \omega \in \mathbb{R}^d, \quad (3)$$

where the polynomial $P \in \mathbb{C}[T]$ is given by

$$P(e^{-i\omega}) = m^{-1} \sum_{\alpha \in \mathbb{Z}^d} p(\alpha) e^{-i\alpha\omega}, \quad \omega \in \mathbb{R}^d.$$

For the characters $\chi \in G'$, we define the so-called isotypical components P_χ of P of weight χ by

$$P_\chi(e^{-i\omega}) = m^{-1} \sum_{\alpha \equiv \chi \pmod{M\mathbb{Z}^d}} p(\alpha) e^{-i\alpha\omega}, \quad \chi \in G'. \quad (4)$$

Clearly, $P = \sum_{\chi \in G'} P_\chi$.

- (ii) One usually assumes that $\widehat{\phi}(0) = 1$ by proper normalization. This assumption on $\widehat{\phi}$ and (3) allows us to read all properties of ϕ from the polynomial P , since the refinement equation (3) then implies

$$\widehat{\phi}(\omega) = \prod_{\ell=1}^{\infty} P(e^{-i(M^T)^{-\ell}\omega}), \quad \omega \in \mathbb{R}^d.$$

The uniform convergence of this infinite product on compact sets is guaranteed by $P(\mathbf{1}) = 1$.

- (iii) One of the approximation properties of ϕ is the requirement that the translates $T_\alpha \phi$, $\alpha \in \mathbb{Z}^d$, form a partition of unity. Then

$$P_\chi(\mathbf{1}) = m^{-1}, \quad \chi \in G'. \quad (5)$$

The real-valued functions ψ_j , $j = 1, \dots, N$, are assumed to be of the form

$$\widehat{\psi}_j(M^T \omega) = Q_j(e^{-i\omega}) \widehat{\phi}(\omega),$$

where $Q_j \in \mathbb{C}[T]$ have real coefficients. These assumptions imply that ψ_j have compact support and, as in (2), are finite linear combination of $U_M T_\alpha \phi$.

We next describe the method called UEP (unitary extension principle) that allows us to determine the polynomials Q_j , $1 \leq j \leq N$, such that the family Ψ as in Definition 2.1 is a wavelet tight frame of $L_2(\mathbb{R}^d)$, see [17, 34]. To that purpose we need to introduce additional notation. Note that the group G acts on the coordinate ring $\mathbb{C}[T]$ by

$$P \mapsto P^\sigma(e^{-i\omega}) := P(e^{-i(\omega+\sigma)}), \quad \sigma \in G, \quad \omega \in \mathbb{R}^d,$$

and this operation commutes with the $*$ -involution.

Theorem 2.2. (UEP) *Let the trigonometric polynomial $P \in \mathbb{C}[T]$ satisfy $P(1) = 1$. If the trigonometric polynomials $Q_j \in \mathbb{C}[T]$, $1 \leq j \leq N$, satisfy the identities*

$$\delta_{\sigma,\tau} - P^{\sigma*}(e^{-i\omega})P^\tau(e^{-i\omega}) = \sum_{j=1}^N Q_j^{\sigma*}(e^{-i\omega})Q_j^\tau(e^{-i\omega}), \quad \sigma, \tau \in G, \quad (6)$$

then the family Ψ is a wavelet tight frame of $L_2(\mathbb{R}^d)$.

Certainly, the necessary condition for such Q_j to exist is that the polynomial $1 - P^*P \in \mathbb{R}[T]$ is non-negative for all $\omega \in \mathbb{R}^d$. We give an equivalent formulation of the identities in Theorem 2.2.

Theorem 2.3. *The identities (6) are equivalent to*

$$U(e^{-i\omega})^*U(e^{-i\omega}) = I_m, \quad \omega \in \mathbb{R}^d, \quad (7)$$

with

$$U^* = [P^{\sigma*} \quad Q_1^{\sigma*} \quad \cdots \quad Q_N^{\sigma*}]_{\sigma \in G} \in M_{m \times (N+1)}(\mathbb{C}[T]).$$

Moreover, this implies the “sub-QMF” condition

$$1 - \sum_{\sigma \in G} P^\sigma(e^{-i\omega})P^{\sigma*}(e^{-i\omega}) \geq 0 \quad \text{for all } \omega \in \mathbb{R}^d. \quad (8)$$

We note that (8) implies (iii). Moreover, the results in [32], see Theorem 2.4, imply that finding the sum of squares decomposition of $f := 1 - \sum_{\sigma \in G} P^\sigma P^{\sigma*}$, $f \in \mathbb{R}[T]$, is sufficient for the existence of the polynomials Q_j in Theorem 2.2. The authors in [32] also provide a method for the construction of Q_j from the sum of squares decomposition of the trigonometric polynomial f .

Theorem 2.4. *If the polynomial $P \in \mathbb{C}[T]$ and the matrix-valued polynomial $H \in M_{1 \times s}(\mathbb{C}[T])$ exist and satisfy*

$$\sum_{\sigma \in G} P^\sigma(e^{-i\omega})P^{\sigma*}(e^{-i\omega}) + H(e^{-iM^T\omega})H^*(e^{-iM^T\omega}) = 1, \quad (9)$$

then there exist trigonometric polynomials $Q_1, \dots, Q_N \in \mathbb{C}[T]$ with real coefficients, $N \leq m+s$, that satisfy (6).

Remark 2.5. Let χ_j be the elements of $G' = \{\chi_1, \dots, \chi_m\}$ and H_j be the columns of $H = (H_1, \dots, H_s) \in M_{1 \times s}(\mathbb{C}[T])$. The proof of Theorem 2.4 in [32] yields the explicit form of Q_1, \dots, Q_N , namely

$$Q_j(e^{-i\omega}) = m^{-1/2}e^{-i\omega\chi_j}(1 - mP(e^{-i\omega})P_{\chi_j}^*(e^{-i\omega})), \quad 1 \leq j \leq m, \quad (10)$$

$$Q_{m+j}(e^{-i\omega}) = P(e^{-i\omega})H_j(e^{-iM^T\omega}), \quad 1 \leq j \leq s. \quad (11)$$

3 Equivalent formulations of UEP and moment problems

In this section we give several equivalent formulations of the Unitary Extension Principle stated in Theorem 2.2 that allow us to establish a connection between UEP and moment problems. Firstly, we reformulate UEP in terms of the isotypical components $P_\chi, Q_{j,\chi}$ of the polynomials P, Q_j . By $\langle \sigma, \chi \rangle = e^{i\sigma \cdot \chi}$, $\sigma \in G$ and $\chi \in G'$, we denote the natural pairing between G and G' . Note that $\langle \sigma, \chi \rangle$ is a root of unity of order dividing $|G|$.

Theorem 3.1. *Let $P \in \mathbb{C}[T]$ satisfy $P(\mathbf{1}) = 1$ and $P_\chi(\mathbf{1}) = m^{-1}$. If $Q_j \in \mathbb{C}[T]$, $1 \leq j \leq N$, satisfy the identities*

$$\begin{aligned} P_\chi^* P_\chi + \sum_{j=1}^N Q_{j,\chi}^* Q_{j,\chi} &= m^{-1}, \quad \chi \in G', \\ P_\chi^* P_\eta + \sum_{j=1}^N Q_{j,\chi}^* Q_{j,\eta} &= 0 \quad \chi, \eta \in G', \quad \chi \neq \eta, \end{aligned} \tag{12}$$

then the family Ψ is a wavelet tight frame of $L_2(\mathbb{R}^d)$.

Proof. Note that $P = \sum_{\chi \in G'} P_\chi$ and $P_\chi = m^{-1} \sum_{\sigma \in G} \langle \sigma, \chi \rangle P^\sigma$ imply

$$P^* = \sum_{\chi \in G'} P_\chi^* \quad \text{and} \quad P^{\sigma*} = \sum_{\chi \in G'} (P_\chi^*)^\sigma = \sum_{\chi \in G'} \langle \sigma, \chi \rangle P_\chi^*.$$

Thus,

$$P^{\sigma*} P = \sum_{\chi, \eta' \in G'} \langle \sigma, \eta' \rangle P_\chi^* P_{\eta'} = \sum_{\eta \in G'} \langle \sigma, \eta \rangle \sum_{\chi \in G'} \langle \sigma, \chi \rangle P_\chi^* P_{\chi+\eta}$$

for $\eta' = \chi + \eta$. Similarly for Q_j . Therefore, the system in Theorem 2.2 is equivalent to

$$\sum_{\chi \in G'} \langle \sigma, \chi \rangle \left(P_\chi^* P_{\chi+\eta} + \sum_{j=1}^N Q_{j,\chi}^* Q_{j,\chi+\eta} \right) = \delta_{\sigma,0} \delta_{\eta,0}, \quad \eta \in G', \quad \sigma \in G,$$

For fixed $\eta \in G'$, this is a system of m equations indexed by $\sigma \in G$ in m unknowns $P_\chi^* P_{\chi+\eta} + \sum_{j=1}^N Q_{j,\chi}^* Q_{j,\chi+\eta}$, $\chi \in G'$. The corresponding system matrix $A = (\langle \sigma, \chi \rangle)_{\sigma \in G, \chi \in G'}$ is invertible and $A^{-1} = m^{-1} A^*$. Thus, (13) is equivalent to (12). \square

To establish the connection between UEP and moment problems, we rewrite the above result in matrix form. For this purpose we need to introduce additional notation. Let a set \mathcal{I} be a subset of \mathbb{Z}^d containing $\{\alpha \in \mathbb{Z}^d : p(\alpha) \neq 0\}$. We also define the orthogonal projections $E_\chi \in \mathbb{R}^{|\mathcal{I}| \times |\mathcal{I}|}$ to be diagonal matrices with diagonal entries given by

$$E_\chi(\alpha, \alpha) = \begin{cases} 1, & \alpha \equiv \chi \pmod{M\mathbb{Z}^d}, \\ 0, & \text{otherwise,} \end{cases} \quad \alpha \in \mathcal{I}.$$

Theorem 3.2. Let a row vector $\mathbf{p} = m^{-1}[p(\alpha) : \alpha \in \mathcal{I}]$, a column vector $x(\mathbf{e}^{-i\omega}) = [e^{-i\alpha\omega} : \alpha \in \mathcal{I}]^T$ and $P(\mathbf{e}^{-i\omega}) = \mathbf{p} \cdot x(\mathbf{e}^{-i\omega})$ satisfy $P(\mathbf{1}) = 1$, $P_\chi(\mathbf{1}) = m^{-1}$. If $Q_j(\mathbf{e}^{-i\omega}) = \mathbf{q}_j \cdot x(\mathbf{e}^{-i\omega})$ with row vectors \mathbf{q}_j of length $|\mathcal{I}|$, $1 \leq j \leq N$, satisfy the identities

$$x^T(\mathbf{e}^{i\omega}) E_\chi \left(\text{diag}(\mathbf{p}) - \mathbf{p}^T \mathbf{p} - \sum_{j=1}^N \mathbf{q}_j^T \mathbf{q}_j \right) E_\eta x(\mathbf{e}^{-i\omega}) = 0 \quad \chi, \eta \in G', \quad (13)$$

then the family Ψ is a wavelet tight frame of $L_2(\mathbb{R}^d)$.

Proof. Note that the definition of E_χ implies that $P_\chi(\mathbf{e}^{-i\omega}) = \mathbf{p} \cdot E_\chi \cdot x(\mathbf{e}^{-i\omega})$ and $Q_{j,\chi}(\mathbf{e}^{-i\omega}) = \mathbf{q}_j \cdot E_\chi \cdot x(\mathbf{e}^{-i\omega})$, $j = 1, \dots, N$. Due to $P_\chi(\mathbf{1}) = m^{-1}$ we get

$$x^T(\mathbf{e}^{i\omega}) E_\chi \text{diag}(\mathbf{p}) E_\chi x(\mathbf{e}^{-i\omega}) = m^{-1}$$

and

$$x^T(\mathbf{e}^{i\omega}) E_\chi \text{diag}(\mathbf{p}) E_\eta x(\mathbf{e}^{-i\omega}) = 0, \quad \chi \neq \eta.$$

Therefore, (13) is equivalent to (12). \square

Define the matrices

$$R = \text{diag}(\mathbf{p}) - \mathbf{p}^T \mathbf{p} \quad \text{and} \quad S = \sum_{j=1}^N \mathbf{q}_j^T \mathbf{q}_j. \quad (14)$$

Then the task of constructing tight wavelet frames can be formulated as the following problem of **semi-definite programming**: for a given R find a matrix $O \in \mathbb{R}^{|\mathcal{I}| \times |\mathcal{I}|}$ such that

$$x^T(\mathbf{e}^{i\omega}) E_\chi \cdot O \cdot E_\eta x(\mathbf{e}^{-i\omega}) = 0, \quad \chi, \eta \in G', \quad (15)$$

and $R + O$ is positive semi-definite. If such matrix $O \in \mathbb{R}^{|\mathcal{I}| \times |\mathcal{I}|}$ does not exist increase the set \mathcal{I} .

To determine the frame generators, set $S = R + O$ and find \mathbf{q}_j by standard factorization techniques from linear algebra.

Note that the identities (15) are equivalent to the following linear constraints on the null-matrices O

$$\sum_{\alpha \equiv \chi, \beta \equiv \eta} O_{\alpha, \beta} \mathbf{e}^{-i(\beta - \alpha)} = 0, \quad \chi, \eta \in G',$$

or, equivalently,

$$\sum_{\alpha \equiv \chi} O_{\alpha, \alpha + \tau} = 0, \quad \tau \in \{\beta - \alpha : \alpha, \beta \in \mathcal{I}\}.$$

Example 3.3. To illustrate the concept of null-matrices, we consider first a very prominent one-dimensional example of a Daubechies wavelet. Let

$$P(\mathbf{e}^{-i\omega}) = \mathbf{p} \cdot x(\mathbf{e}^{-i\omega}), \quad \mathbf{p} = \frac{1}{8} \begin{bmatrix} 1 + \sqrt{3} & 3 + \sqrt{3} & 3 - \sqrt{3} & 1 - \sqrt{3} \end{bmatrix},$$

and $x(e^{-i\omega}) = [e^{-i\alpha\omega} : \alpha \in \{0, 1, 2, 3\}]^T$. In this case $M = m = 2$, $G = \{0, \pi\}$ and $G' = \{0, 1\}$. Therefore, the orthogonal projections $E_\chi \in \mathbb{R}^{4 \times 4}$, $\chi \in G'$, are given by

$$E_0 = \text{diag}[1, 0, 1, 0] \quad \text{and} \quad E_1 = \text{diag}[0, 1, 0, 1].$$

By (14), we have

$$R = \frac{1}{64} \begin{bmatrix} 4 + 6\sqrt{3} & -6 - 4\sqrt{3} & -2\sqrt{3} & 2 \\ -6 - 4\sqrt{3} & 12 + 2\sqrt{3} & -6 & 2\sqrt{3} \\ -2\sqrt{3} & -6 & 12 - 2\sqrt{3} & -6 + 4\sqrt{3} \\ 2 & 2\sqrt{3} & -6 + 4\sqrt{3} & 4 - 6\sqrt{3} \end{bmatrix},$$

which is not positive semi-definite. Define

$$O = \frac{1}{64} \begin{bmatrix} -8\sqrt{3} & 8\sqrt{3} & 0 & 0 \\ 8\sqrt{3} & -8\sqrt{3} & 0 & 0 \\ 0 & 0 & 8\sqrt{3} & -8\sqrt{3} \\ 0 & 0 & -8\sqrt{3} & 8\sqrt{3} \end{bmatrix}$$

satisfying (15). Then $S = R + O$ is positive semi-definite, of rank one, and yields the well-known Daubechies wavelet, see [16] defined by

$$Q_1(e^{-i\omega}) = \frac{1}{8} \begin{bmatrix} 1 - \sqrt{3} & -3 + \sqrt{3} & 3 + \sqrt{3} & -1 - \sqrt{3} \end{bmatrix} \cdot (e^{-i\omega}).$$

Another two-dimensional example of one possible choice of an appropriate null-matrix satisfying (15) is given in Example 4.4.

Another, very similar, way of working with null-matrices was pursued already in [6]. We first sketch the derivation of another equivalent formulation for UEP, see [6, Theorem 2.4], keeping the notation as close as possible to the one in [6]. Multiplying the equations (6) by $m^{-1} \cdot e^{-i\sigma\chi}$ and summing over $\sigma \in G$ we get an equivalent system of equations

$$m^{-1} - P(e^{-i\omega}) e^{i\chi \cdot \omega} \overline{P_\chi(e^{-iM^T\omega})} = \sum_{j=1}^N Q_j(e^{-i\omega}) e^{i\chi \cdot \omega} \overline{Q_{j,\chi}(e^{-iM^T\omega})}, \quad \chi \in G'.$$

This equivalent form of the UEP allows us to use semi-definite programming for tight wavelet frame construction. The null-matrices O are now part of the matrix S in the following result.

Theorem 3.4. *Let the sequences c_χ be given by*

$$\sum_{\alpha \in \mathbb{Z}^d} c_\chi(\alpha) e^{-i\alpha \cdot \omega} := m^{-1} - P(e^{-i\omega}) e^{i\chi \cdot \omega} \overline{P_\chi(e^{-iM^T\omega})}, \quad \chi \in G'.$$

Then the following statements are equivalent:

- (i) *There exist $Q_j \in \mathbb{C}[T]$, $j = 1, \dots, N$, satisfying UEP and whose coordinate degrees are determined by \mathcal{I} .*

(ii) There exists a positive semi-definite matrix $S = (s_{\alpha,\beta})_{\alpha,\beta \in \mathcal{I}}$ such that $(-S, t = 0)$ solves the optimization problem

minimize t

subject to linear constraints

$$c_\chi(\alpha) = \sum_{\beta \in \mathbb{Z}^d} s_{\alpha+\chi+M\beta, \chi+M\beta}, \quad \chi \in G', \quad \alpha \in \{\lambda - \mu : \lambda, \mu \in \mathcal{I}\},$$

$$\text{and} \quad t \cdot I - (-S) \succeq 0.$$

For an application of the method described in Theorem 3.4 see [6].

4 Real algebraic geometry and wavelet frames

In this section we use the result of Theorem 2.4 that allows us to reduce the problem of existence of Q_j in (6) to the existence of an sos decomposition of a single nonnegative polynomial

$$f := 1 - \sum_{\sigma \in G} P^\sigma P^{\sigma*} \in \mathbb{R}[T]. \quad (16)$$

In subsection 4.1, for dimension $d = 2$, we show that the polynomials $Q_1, \dots, Q_N \in \mathbb{C}[T]$ as in Theorem 2.2 always exist. This result is based on recent progress in real algebraic geometry. In subsection 4.2, we give sufficient conditions for the existence of the Q_j 's in the multidimensional case.

We start by deriving an equivalent formulation of the sub-QMF condition (8) that follows from part (a) of the following lemma.

Lemma 4.1. *Let $P \in \mathbb{C}[T]$ with isotypical components $P_\chi, \chi \in G'$.*

$$(a) \quad \sum_{\sigma \in G} P^\sigma P^{\sigma*} = |G| \cdot \sum_{\chi \in G'} P_\chi P_\chi^*.$$

(b) *Given a decomposition*

$$1 - \sum_{\sigma \in G} P^\sigma P^{\sigma*} = \sum_{j=1}^r H_j H_j^* \quad (17)$$

of length r with $H_j \in \mathbb{C}[T]$, then there also exists a decomposition

$$1 - \sum_{\sigma \in G} P^\sigma P^{\sigma*} = \sum_{j=1}^r \sum_{\chi \in G'} \tilde{H}_{j,\chi} \tilde{H}_{j,\chi}^* \quad (18)$$

of length mr with G -invariant matrix polynomials $\tilde{H}_{j,\chi} \in \mathbb{C}[T]$.

Proof. (a) follows from [4, Lemma 2.3]. For (b) we observe that the left-hand side of (17) is G -invariant, i.e.,

$$1 - \sum_{\sigma \in G} P^\sigma P^{\sigma*} = \frac{1}{|G|} \sum_{j=1}^r \sum_{\sigma \in G} H_j^\sigma H_j^{\sigma*}.$$

Using the result in (a) we get that the left-hand side of (17) is equal to

$$\frac{1}{|G|} \sum_{j=1}^r \sum_{\sigma \in G} H_j^\sigma H_j^{\sigma*} = \sum_{j=1}^r \sum_{\chi \in G'} H_{j,\chi} H_{j,\chi}^*.$$

For every $\chi \in G'$ choose a lift $\alpha_\chi \in \mathbb{Z}^d$ with respect to the natural surjection $\mathbb{Z}^d \rightarrow G'$, $\alpha \mapsto \chi_\alpha$, given by

$$\chi_\alpha(\sigma) = \sigma_1^{\alpha_1} \cdots \sigma_d^{\alpha_d}, \quad \sigma \in G, \quad \alpha \in \mathbb{Z}^d.$$

Let $\tilde{H}_{j,\chi} := z^{-\alpha_\chi} H_{j,\chi}$. Then, for all j and χ , the polynomial $\tilde{H}_{j,\chi}$ is G -invariant and satisfies $\tilde{H}_{j,\chi} \tilde{H}_{j,\chi}^* = H_{j,\chi} H_{j,\chi}^*$. \square

4.1 Scalar 2-dimensional case

In this section we show that in the two-dimensional case ($d = 2$) the question of existence of a wavelet tight frame can be positively answered using the results from [35]. Thus, the main result of this section answers a long standing open question for the existence of tight wavelet frames. We also show that the result of Lemma 4.1, combined with Theorem 2.4, leads to an elegant method for the construction of tight frames.

Theorem 4.2. *Let $d = 2$, $P \in \mathbb{C}[T]$ be a polynomial satisfying $P(\mathbf{1}) = 1$ and $\sum_{\sigma \in G} |P^\sigma|^2 \leq 1$ on $\mathbb{T}^2 = T(\mathbb{R})$. Then there exist $N \in \mathbb{N}$ and polynomials $Q_1, \dots, Q_N \in \mathbb{C}[T]$ that satisfy*

$$\delta_{\sigma,\tau} = P^{\sigma*} P^\tau + \sum_{j=1}^N Q_j^{\sigma*} Q_j^\tau, \quad \sigma, \tau \in G. \quad (19)$$

Proof. The torus T is a non-singular affine algebraic surface over \mathbb{R} , and $T(\mathbb{R})$ is compact. The polynomial f in (16) is in $\mathbb{R}[T]$ and is nonnegative on $T(\mathbb{R})$ by assumption. By Corollary 3.4 of [35], there exist $H_1, \dots, H_r \in \mathbb{C}[T]$ satisfying $f = \sum_{j=1}^r |H_j|^2$. According to Lemma 4.1 part (b), the polynomials H_j can be taken to be G -invariant. Thus, by Theorem 2.4, there exist polynomials Q_1, \dots, Q_N satisfying (19). \square

Lemma 4.1 part (a) sometimes yields an elegant method for the construction of H_j . We have

$$f = 1 - \sum_{\sigma \in G} |P^\sigma|^2 = 1 - m \sum_{\chi \in G'} |P_\chi|^2 = m \sum_{\chi \in G'} \left(\frac{1}{m^2} - |P_\chi|^2 \right).$$

So it suffices to find an sos decomposition for each of the polynomials $m^{-2} - |P_\chi|^2$, provided that they are all nonnegative. This nonnegativity assumption is satisfied, for example, for the special case when all coefficients $p(\alpha)$ of P are nonnegative. This is due to the simple fact that for nonnegative $p(\alpha)$ we get

$$|P_\chi|^2 \leq |P_\chi(\mathbf{1})|^2 = m^{-2}$$

on \mathbb{T}^d , for all $\chi \in G'$.

Example 4.3. As an illustration of the case of nonnegative coefficients we consider the three-directional piecewise linear box spline with the symbol

$$P(z_1, z_2) = \frac{1}{8} (1 + z_1)(1 + z_2)(1 + z_1 z_2), \quad z_j = e^{-i\omega_j}.$$

The sos decomposition for the isotypical components yields

$$f = 1 - m \sum_{\chi \in G'} |P_\chi|^2 = \frac{1}{4} \sin^2(\omega_1) + \frac{1}{4} \sin^2(\omega_2) + \frac{1}{4} \sin^2(\omega_1 + \omega_2).$$

Thus, in (17) we have a decomposition with $N = 3$. Since each of H_1, H_2, H_3 has only one isotypical component, we get a representation $f = \tilde{H}_1^2 + \tilde{H}_2^2 + \tilde{H}_3^2$ with 3 G -invariant polynomials \tilde{H}_j . By Theorem 2.4 we get 7 frame generators. Note that the method in [32, Example 2.4] yields 6 generators, one of slightly larger support. The method in [8, Section 4] based on properties of the Kronecker product leads to 7 frame generators whose support is the same as the one of P . One can also employ the technique discussed in [21, Section] and get 7 frame generators.

Another prominent example from bivariate subdivision is the so-called butterfly scheme. This example shows that the real advantage of treating the isotypical components of P separately is when one works with P of larger support.

Example 4.4. The butterfly scheme describes an interpolatory subdivision method which is used in order to generate a smooth regular surface interpolating a given set of points [20]. The trigonometric polynomial P associated with the butterfly scheme is

$$\begin{aligned} P(z_1, z_2) = & \frac{1}{4} + \frac{1}{8} \left(z_1 + z_2 + z_1 z_2 + z_1^{-1} + z_2^{-1} + z_1^{-1} z_2^{-1} \right) \\ & + \frac{1}{32} \left(z_1^2 z_2 + z_1 z_2^2 + z_1 z_2^{-1} + z_1^{-1} z_2 + z_1^{-2} z_2^{-1} + z_1^{-1} z_2^{-2} \right) \\ & - \frac{1}{64} \left(z_1^3 z_2 + z_1^3 z_2^2 + z_1^2 z_2^3 + z_1 z_2^3 + z_1^2 z_2^{-1} + z_1 z_2^{-2} \right. \\ & \left. + z_1^{-1} z_2^2 + z_1^{-2} z_2 + z_1^{-3} z_2^{-1} + z_1^{-3} z_2^{-2} + z_1^{-2} z_2^{-3} + z_1^{-1} z_2^{-3} \right). \end{aligned}$$

Its first isotypical component is $P_{0,0} = \frac{1}{4}$, which is the case for every interpolatory subdivision scheme. The other isotypical components, in terms of $z_k = e^{-i\omega_k}$ ($k = 1, 2$) are $P_{1,0}(z_1, z_2) = \frac{1}{4} \cos(\omega_1) + \frac{1}{16} \cos(\omega_1 + 2\omega_2) - \frac{1}{32} \cos(3\omega_1 + 2\omega_2) - \frac{1}{32} \cos(\omega_1 - 2\omega_2)$, i.e.,

$$P_{1,0}(z_1, z_2) = \frac{1}{4} \cos(\omega_1) + \frac{1}{8} \sin^2(\omega_1) \cos(\omega_1 + 2\omega_2),$$

and $P_{0,1}(z_1, z_2) = P_{1,0}(z_2, z_1)$, $P_{1,1}(z_1, z_2) = P_{1,0}(z_1 z_2, z_2^{-1})$. Note that $|P_\chi| \leq \frac{1}{4}$ for all $\chi \in G'$. Elementary computation shows that $1 - 16 |P_{1,0}(z_1, z_2)|^2 = 1 - \cos^2(\omega_1) - \cos(\omega_1) \sin^2(\omega_1) \cos(\omega_1 + 2\omega_2) - \frac{1}{4} \sin^4(\omega_1) \cos^2(\omega_1 + 2\omega_2)$. Setting $u_j := \sin(\omega_j)$, $j = 1, 2$, $v := \sin(\omega_1 + \omega_2)$, $v' := \sin(\omega_1 - \omega_2)$, $w := \sin(\omega_1 + 2\omega_2)$ and $w' := \sin(2\omega_1 + \omega_2)$, we get

$$1 - 16 |P_{1,0}(z_1, z_2)|^2 = \frac{1}{4} u_1^2 \left(w^2 + (u_2^2 + v^2)^2 + 2u_2^2 + 2v^2 \right).$$

Therefore,

$$\begin{aligned} 1 - \sum_{\sigma \in G} |P^\sigma|^2 = & \frac{1}{4} \left(u_1^2 u_2^2 + u_1^2 v^2 + u_2^2 v^2 \right) + \frac{1}{16} \left(u_1^2 w^2 + u_2^2 w'^2 + v^2 v'^2 \right) \\ & + \frac{1}{16} \left(u_1^2 (u_2^2 + v^2)^2 + u_2^2 (u_1^2 + v^2)^2 + v^2 (u_1^2 + u_2^2)^2 \right). \end{aligned}$$

This provides a decomposition $1 - \sum_{\sigma \in G} |P^\sigma|^2 = \sum_{j=1}^9 |H_j|^2$ into a sum of 9 squares. As in the previous example, each H_j has only one nonzero isotypical component H_{j,χ_j} . Thus, part (b) of Lemma 4.1 and by Theorem 2.4, there exists a tight frame with 13 generators. Namely, by Remark 2.5, we get

$$\begin{aligned} Q_1(z_1, z_2) &= \frac{1}{2} - \frac{1}{2}P(z_1, z_2), & Q_2(z_1, z_2) &= \frac{1}{2}z_1 - 2P(z_1, z_2)P_{(1,0)}^*(z_1, z_2) \\ Q_3(z_1, z_2) &= Q_2(z_2, z_1), & Q_4(z_1, z_2) &= Q_2(z_1z_2, z_2^{-1}) \\ Q_{4+j}(z_1, z_2) &= P(z_1, z_2)\tilde{H}_{j,\chi_j}, & j &= 1, \dots, 9, \end{aligned}$$

where \tilde{H}_{j,χ_j} are the lifted isotypical components defined as in Lemma 4.1. Let $\mathcal{I} = \{0, \dots, 7\}^2$, $P(z_1, z_2) = \mathbf{p} \cdot x(z_1, z_2)$ and $Q_j(z_1, z_2) = \mathbf{q}_j \cdot x(z_1, z_2)$ with $x(z_1, z_2) = [z^\alpha : \alpha \in \mathcal{I}]^T$. The corresponding null-matrix $O \in \mathbb{R}^{64 \times 64}$ defined in section 3 is given by

$$x^T(z_1^{-1}, z_2^{-1}) \cdot O \cdot x(z_1, z_2) = x^T(z_1^{-1}, z_2^{-1}) \left[\sum_{j=1}^{13} \mathbf{q}_j^T \mathbf{q}_j - \text{diag}(\mathbf{p}) + \mathbf{p}^T \mathbf{p} \right] x(z_1, z_2).$$

Note that other factorizations of the positive semi-definite matrix $\text{diag}(\mathbf{p}) - \mathbf{p}^T \mathbf{p} + O$ of rank 13 lead to other possible tight frames with at least 13 frame generators. An advantage of using semi-definite programming techniques is that it can possibly yield Q_j of smaller degree and reduce the rank of $\text{diag}(\mathbf{p}) - \mathbf{p}^T \mathbf{p} + O$.

Using the technique of semi-definite programming the authors in [6] constructed numerically a tight frame for the butterfly scheme with 18 frame generators. The advantage of our present construction is that the frame generators are determined analytically. The disadvantage is that their support is approximately twice as large as that of the frame generators in [6].

The next example is one of the family of interpolatory $\sqrt{3}$ -subdivision studied in [27]. The associated dilation matrix is $M = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$ and $m = 3$.

Example 4.5. The symbol of the scheme is given by

$$P(z_1, z_2) = P_{(0,0)}(z_1, z_2) + P_{(1,0)}(z_1, z_2) + P_{(0,1)}(z_1, z_2)$$

with isotypical components $P_{(0,0)} = \frac{1}{3}$,

$$P_{(0,1)}(z_1, z_2) = \frac{4}{27}(z_2 + z_1^{-1} + z_1z_2^{-1}) - \frac{1}{27}(z_1^{-2}z_2^2 + z_1^2 + z_2^{-2})$$

and $P_{(1,0)}(z_1, z_2) = P_{(0,1)}(z_2, z_1)$. We have by Lemma 4.1 and due to $|P_{(0,1)}|^2 = |P_{(1,0)}|^2$

$$1 - \sum_{\sigma \in G} |P^\sigma|^2 = 2 \left(\frac{1}{9} - |P_{(0,1)}|^2 \right),$$

thus it suffices to consider only

$$\begin{aligned} m^{-2} &- |P_{0,1}|^2 = 3^{-2} - 27^{-2} \left(51 + 16 \cos(\omega_1 + \omega_2) + 16 \cos(2\omega_1 - \omega_2) \right. \\ &+ 16 \cos(\omega_1 - 2\omega_2) + 2 \cos(2\omega_1 + 2\omega_2) + 2 \cos(2\omega_1 - 4\omega_2) + 2 \cos(4\omega_1 - 2\omega_2) \\ &- 8 \cos(3\omega_1) - 8 \cos(3\omega_2) - 8 \cos(3\omega_1 - 3\omega_2) \Big). \end{aligned}$$

Numerical tests show that this polynomial is nonnegative.

4.2 Scalar multivariate case

In the general multivariate case $d \geq 2$, in Theorem 4.7, we provide a sufficient condition for the existence of a sums of squares decomposition of f in (16). This condition is based on the properties of the Hessian of $f \in \mathbb{R}[T]$

$$\text{Hess}(f)(e^{-i\omega}) = (D^\mu f(e^{-i\omega}))_{\mu \in \mathbb{N}_0^d, |\mu|=2},$$

where D^μ denotes the $|\mu|$ -th partial derivative with respect to $\omega \in \mathbb{R}^d$.

Theorem 4.6. *Let V be a non-singular affine \mathbb{R} -variety for which $V(\mathbb{R})$ is compact, and let $f \in \mathbb{R}[V]$ with $f \geq 0$ on $V(\mathbb{R})$. For every $\xi \in V(\mathbb{R})$ with $f(\xi) = 0$, assume that the Hessian of f at ξ is strictly positive definite. Then f is a sum of squares in $\mathbb{R}[V]$.*

Proof. The hypotheses imply that f has only finitely many zeros in $V(\mathbb{R})$. Therefore the claim follows from [36], Corollary 2.17 and Example 3.18. \square

Theorem 4.6 implies the following result.

Theorem 4.7. *Let $P \in \mathbb{C}[T]$, and write $f = 1 - \sum_{\sigma \in G} |P^\sigma|^2$. Assume that $P(\mathbf{1}) = 1$ and $f \geq 0$ on $T(\mathbb{R}) = \mathbb{T}^d$. If the Hessian of f is positive definite at any zero of f in \mathbb{T}^d , there exist $N \in \mathbb{N}$ and polynomials $Q_1, \dots, Q_N \in \mathbb{C}[T]$ satisfying (6).*

Proof. By Theorem 4.6, f is a sum of squares in $\mathbb{R}[T]$. The claim follows then by Theorem 4.2. \square

Due to $P(\mathbf{1}) = 1$, $z = \mathbf{1}$ is obviously a zero of f . We show next how to express the Hessian of f at $\mathbf{1}$ in terms of the gradient $\nabla P(\mathbf{1})$ and the Hessian of P at $\mathbf{1}$, if P additionally satisfies the so-called sum rules of order 2, or, equivalently, satisfies the zero conditions of order 2. We say that $P \in \mathbb{C}[T]$ satisfies zero conditions of order k , if

$$D^\mu P(e^{-i\sigma}) = 0, \quad \mu \in \mathbb{N}_0^d, \quad |\mu| < k, \quad \sigma \in G \setminus \{0\},$$

see [25, 26] for details. The assumption that P satisfies sum rules of order 2 together with $P(\mathbf{1}) = 1$ are necessary for the continuity of the corresponding refinable function ϕ .

Lemma 4.8. *Let $P \in \mathbb{C}[T]$ with real coefficients satisfy the sum rules of order 2 and $P(\mathbf{1}) = 1$. Then the Hessian of $f = 1 - \sum_{\sigma \in G} |P^\sigma|^2$ at $\mathbf{1}$ is equal to*

$$-2\text{Hess}(P)(\mathbf{1}) - 2\nabla P(\mathbf{1})^* \nabla P(\mathbf{1}).$$

Proof. We expand the trigonometric polynomial $P : \mathbb{T}^d \rightarrow \mathbb{C}$ in a neighborhood of $\mathbf{1}$ and get

$$P(e^{-i\omega}) = 1 + \nabla P(\mathbf{1})\omega + \frac{1}{2}\omega^T \text{Hess}(P)(\mathbf{1})\omega + \mathcal{O}(|\omega|^3).$$

Note that, since the coefficients of P are real, the row vector $v = \nabla P(\mathbf{1})$ is purely imaginary and $\text{Hess}(P)(\mathbf{1})$ is real and symmetric. The sum rules of order 2 are equivalent to

$$P^\sigma(\mathbf{1}) = 0, \quad \nabla P^\sigma(\mathbf{1}) = 0 \quad \text{for all } \sigma \in G \setminus \{0\}.$$

Thus, we have $P^\sigma(e^{-i\omega}) = \mathcal{O}(|\omega|^2)$ for all $\sigma \in G \setminus \{0\}$. Simple computation yields

$$\begin{aligned} |P(e^{-i\omega})|^2 &= 1 + (v + \bar{v})\omega + \omega^T (\text{Hess}(P)(\mathbf{1}) + v^* v) \omega + \mathcal{O}(|\omega|^3) \\ &= 1 + \omega^T (\text{Hess}(P)(\mathbf{1}) + v^* v) \omega + \mathcal{O}(|\omega|^3). \end{aligned}$$

Thus, the claim follows. \square

Remark 4.9. Note that $\text{Hess}(f)$ is a zero matrix, if P is a symbol of interpolatory subdivision schemes, i.e.,

$$P(e^{-i\omega}) = m^{-1} + m^{-1} \sum_{\chi \in G' \setminus \{0\}} P_{\chi}(e^{-i\omega}), \quad \omega \in \mathbb{R}^d,$$

and satisfies zero conditions of order at least 3. This property of $\text{Hess}(f)$ follows directly from the equivalent formulation of zero conditions of order k , see [2]. The examples of P with such properties are for example the butterfly scheme in Example 4.4 and the three-dimensional interpolatory scheme in Example 4.12.

For simplicity of presentation, we start by applying the result of Theorem 4.7 to the 2-dimensional polynomial f from Example 4.3. This example also motivates the statements of Remark 4.11.

Example 4.10. The three-directional box-spline from Example 4.3 is defined by the trigonometric polynomial

$$P(e^{-i\omega}) = e^{-i(\omega_1 + \omega_2)} \cos\left(\frac{\omega_1}{2}\right) \cos\left(\frac{\omega_2}{2}\right) \cos\left(\frac{\omega_1 + \omega_2}{2}\right), \quad \omega \in \mathbb{R}^2.$$

Note that

$$\cos\left(\frac{\omega_1}{2}\right) \cos\left(\frac{\omega_2}{2}\right) \cos\left(\frac{\omega_1 + \omega_2}{2}\right) = 1 - \frac{1}{8} \omega^T \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \omega + \mathcal{O}(|\omega|^4).$$

Therefore, as the trigonometric polynomial P satisfies sum rules of order 2, we get

$$f(e^{-i\omega}) = \frac{1}{8} \omega^T \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \omega + \mathcal{O}(|\omega|^4).$$

Thus, the Hessian of f at $\mathbf{1}$ is positive definite.

To determine other zeroes of f , by Lemma 4.1 part (a), we can use either one of the representations

$$\begin{aligned} f(e^{-i\omega}) &= 1 - \sum_{\sigma \in \{0, \pi\}^2} \prod_{\theta \in \{0, 1\}^2 \setminus \{0\}} \cos^2\left(\frac{(\omega + \sigma) \cdot \theta}{2}\right) \\ &= \frac{1}{4} \sum_{\chi \in \{0, 1\}^2} (1 - \cos^2(\omega \cdot \chi)). \end{aligned}$$

It follows that the zeros of f are all the points $\omega \in \pi\mathbb{Z}^2$ and, by periodicity of f with period π in both coordinate directions, we get that

$$\text{Hess}(f)(e^{-i\omega}) = \text{Hess}(f)(\mathbf{1}), \quad \omega \in \pi\mathbb{Z}^2,$$

is positive definite at all zeros of f .

Remark 4.11. (i) The result of [5, Theorem 2.4] implies the existence of tight frames for multivariate box-splines. According to the notation in [18, p. 127], the corresponding trigonometric polynomial is given by

$$P(e^{-i\omega}) = \prod_{j=1}^n \frac{1 + e^{-i\omega \cdot \xi^{(j)}}}{2}, \quad \omega \in \mathbb{R}^d,$$

where $\Xi = (\xi^{(1)}, \dots, \xi^{(n)}) \in \mathbb{Z}^{d \times n}$ is unimodular and has rank d . (Unimodularity means that all $d \times d$ -submatrices have determinant 0, 1, or -1 .) Moreover, Ξ has the property that leaving out any column $\xi^{(j)}$ does not reduce its rank. (This property guarantees continuity of the box-spline and that the corresponding polynomial P satisfies at least sum rules of order 2.) Then one can show that

$$f(e^{-i\omega}) = 1 - \sum_{\sigma \in G} |P^\sigma(e^{-i\omega})|^2 \geq 0,$$

and the only zeros occur at $\omega \in \pi\mathbb{Z}^d$ and the Hessian of f at these zeros is positive definite. This yields an alternative proof for [5, Theorem 2.4] in the case of box-splines.

(ii) If the summands $m^{-2} - |P_\chi(e^{-i\omega})|^2$ are non-negative for all $\omega \in \mathbb{R}^d$, then it can be easier to determine the zeros of f by determining the common zeros of all of these polynomials.

Example 4.12. There was an attempt to define an interpolatory scheme for 3D-subdivision with dilation matrix $2I_3$ in [14]. There are several inconsistencies in this paper and we give a correct description of the trigonometric polynomial P , the so-called subdivision mask. Note that the scheme we present is an extension of the 2-D butterfly scheme to 3-D data in the following sense: if the data are constant along one of the coordinate directions (or along the main diagonal in \mathbb{R}^3), then the subdivision procedure keeps this property and is identical with the 2-D butterfly scheme.

We describe the trigonometric polynomial P associated with this 3-D scheme by defining its isotypical components. The isotypical components, in terms of $z_k = e^{-i\omega_k}$, $k = 1, 2$, are given by

$$P_{0,0,0}(z_1, z_2, z_3) = 1/8,$$

$$P_{1,0,0}(z_1, z_2, z_3) = \frac{1}{8} \cos \omega_1 + \frac{\lambda}{4} \left(\cos(\omega_1 + 2\omega_2) + \cos(\omega_1 + 2\omega_3) + \cos(\omega_1 + 2\omega_2 + 2\omega_3) \right) - \frac{\lambda}{4} \left(\cos(\omega_1 - 2\omega_2) + \cos(\omega_1 - 2\omega_3) + \cos(3\omega_1 + 2\omega_2 + 2\omega_3) \right),$$

$$P_{0,1,0}(z_1, z_2, z_3) = P_{1,0,0}(z_2, z_1, z_3), \quad P_{0,0,1}(z_1, z_2, z_3) = P_{1,0,0}(z_3, z_1, z_2),$$

$$P_{1,1,1}(z_1, z_2, z_3) = P_{1,0,0}(z_1 z_2 z_3, z_1^{-1}, z_2^{-1}),$$

$$P_{1,1,0}(z_1, z_2, z_3) = \left(\frac{1}{8} - \lambda \right) \cos(\omega_1 + \omega_2) + \lambda \left(\cos(\omega_1 - \omega_2) + \cos(\omega_1 + \omega_2 + 2\omega_3) \right) - \frac{\lambda}{4} \left(\cos(\omega_1 - \omega_2 + 2\omega_3) + \cos(\omega_1 - \omega_2 - 2\omega_3) + \cos(3\omega_1 + \omega_2 + 2\omega_3) + \cos(\omega_1 + 3\omega_2 + 2\omega_3) \right),$$

$$P_{1,0,1}(z_1, z_2, z_3) = P_{1,1,0}(z_1, z_3, z_2), \quad P_{0,1,1}(z_1, z_2, z_3) = P_{1,0,0}(z_2, z_3, z_1),$$

where λ is the so-called tension parameter.

The polynomial P also satisfies

$$P(z_1, z_2, z_3) = \frac{1}{8} (1 + z_1)(1 + z_2)(1 + z_3)(1 + z_1 z_2 z_3) Q(z_1, z_2, z_3), \quad Q(\mathbf{1}) = 1,$$

which implies sum rules of order 2.

- a) For $\lambda = 0$, we have $Q(z_1, z_2, z_3) = 1/(z_1 z_2 z_3)$. Hence, P is the scaling symbol of the trivariate box-spline with direction set $(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)$ whose support center is shifted to the origin.
- b) For $0 \leq \lambda < 1/16$, the subdivision scheme converges and has a continuous limit function. The only zeros of the associated non-negative trigonometric polynomial f are at $\pi\mathbb{Z}^3$, and the Hessian of f at these zeros is given by

$$\text{Hess}(f)(\mathbf{1}) = \text{Hess}(f)(e^{-i\omega}) = \begin{pmatrix} 1 - 16\lambda & \frac{1}{2} - 8\lambda & \frac{1}{2} - 8\lambda \\ \frac{1}{2} - 8\lambda & 1 - 16\lambda & \frac{1}{2} - 8\lambda \\ \frac{1}{2} - 8\lambda & \frac{1}{2} - 8\lambda & 1 - 16\lambda \end{pmatrix}, \quad \omega \in \pi\mathbb{Z}^3.$$

The existence of the sos-decomposition of f is guaranteed by Theorem 4.7; our explicit decomposition is computed as follows.

- b₁) Denote $u := \cos(\omega_1 + \omega_2)$, $v := \cos(\omega_1 + \omega_3)$, $w := \cos(\omega_2 + \omega_3)$, and $\tilde{u} := \sin(\omega_1 + \omega_2)$, $\tilde{v} := \sin(\omega_1 + \omega_3)$, $\tilde{w} := \sin(\omega_2 + \omega_3)$. Elementary computations give

$$P_{1,1,0} = \frac{1}{8} - (1 - u)\left(\frac{1}{8} - \lambda v^2 - \lambda w^2\right) - \lambda(v - w)^2,$$

and

$$\frac{1}{64} - |P_{1,1,0}|^2 = \lambda^2(v^2 - w^2)^2 + \left(\left(\frac{1}{16} - \lambda v^2\right) + \left(\frac{1}{16} - \lambda w^2\right)\right)\left(\frac{1}{8}\tilde{u}^2 + \lambda(v - uw)^2 + \lambda(w - uv)^2\right),$$

which is sos with 7 summands H_j , and each H_j has only one nonzero isotypical component.

- b₂) The isotypical component $P_{1,0,0}$ is not bounded by $1/8$; consider, for example, $P_{1,0,0}(e^{-i\omega})$ at the point $\omega = (-\frac{\pi}{6}, -\frac{2\pi}{3}, -\frac{2\pi}{3})$. Yet we obtain by elementary computations

$$P_{1,0,0} = \frac{1}{8} \cos \omega_1 + \frac{\lambda}{2} A \sin \omega_1, \quad A := \sin 2(\omega_1 + \omega_2 + \omega_3) - \sin 2\omega_2 - \sin 2\omega_3,$$

and

$$\frac{1}{16} - |P_{1,0,0}|^2 - |P_{0,1,0}|^2 - |P_{0,0,1}|^2 - |P_{1,1,1}|^2 = E_{1,0,0} + E_{0,1,0} + E_{0,0,1} + E_{1,1,1},$$

where

$$E_{1,0,0} = \frac{3\lambda}{16} \sin^4 \omega_1 + \frac{\lambda}{64} (2 \sin \omega_1 - A \cos \omega_1)^2 + \frac{1 - 16\lambda}{64} \sin^2 \omega_1 (1 + \lambda A^2);$$

the other $E_{i,j,k}$ are given by the same coordinate transformations as $P_{i,j,k}$. Hence, we obtain an sos with 12 summands H_j , each with only one nonzero isotypical component.

Thus, with the result of Theorem 2.4, we have explicitly constructed a tight frame with 41 generators for the trivariate interpolatory subdivision scheme with tension parameter $0 \leq \lambda < 1/16$.

- c) For $\lambda = 1/16$, the sum rules of order 4 are satisfied. In this particular case, the scheme is C^1 and the Hessian of f at $\mathbf{1}$ is the zero-matrix, thus the result of Theorem 4.7 is not applicable. Nevertheless, the sos decomposition of $1 - \sum |P^\sigma|^2$ in b), with further simplifications for $\lambda = 1/16$, gives a tight frame with 31 generators for the trivariate interpolatory subdivision scheme. Further properties of this scheme will be studied in [7].

The question may arise if there exists a trigonometric polynomial P that satisfies sum rules of a certain order and also satisfies the sub-QMF condition $\sum_{\sigma \in G} |P^\sigma|^2 \leq 1$, but there exists no UEP tight frame as in Theorem 2.2. In other words, can we find P as above such that the non-negative trigonometric polynomial $1 - |P|^2$ is not an sos of trigonometric polynomials? Recall that such polynomials cannot exist in the 1-D and 2-D case.

Theorem 4.13. *There exists $P \in \mathbb{C}[T]$ on the 3-D torus that satisfies sum rules of order 6 and the sub-QMF condition, such that $1 - |P|^2$ is not sos in $\mathbb{R}[T]$.*

The proof is constructive. The following example defines a family of trigonometric polynomials with the properties in Theorem 4.13. We make use of the following local-global result: if the Taylor expansion of $f \in \mathbb{R}[T]$ at one of its roots has, in local coordinates, a homogeneous part of lowest degree which is not sos of real algebraic polynomials, then f is not sos in $\mathbb{R}[T]$.

Example 4.14. Denote $z_j = e^{-i\omega_j}$, $j = 1, 2, 3$. We let

$$P(z) = (1 - cM(z))A(z), \quad z \in T, \quad 0 < c \leq \frac{1}{3},$$

where

$$M(z) = y_1^4 y_2^2 + y_1^2 y_2^4 - y_3^6 - 3y_1^2 y_2^2 y_3^2 \in \mathbb{R}[T], \quad y_j = \sin \omega_j.$$

In the local coordinates (y_1, y_2, y_3) at $z = \mathbf{1}$, M is the well-known Motzkin polynomial in $\mathbb{R}[y_1, y_2, y_3]$; i.e. M is not sos in $\mathbb{R}[y_1, y_2, y_3]$. Moreover, $A \in \mathbb{R}[T]$ is chosen such that

$$D^\alpha A(\mathbf{1}) = \delta_{0,\alpha}, \quad D^\alpha A(\sigma) = 0, \quad 0 \leq |\alpha| < 8, \quad \sigma \in G \setminus \{\mathbf{1}\}, \quad (20)$$

and $\sum_\sigma |A^\sigma|^2 \leq 1$. Such A can be, for example, any scaling symbol of a 3-D orthonormal wavelet with 8 vanishing moments; in particular, the tensor product Daubechies symbol $A(z) = m_8(z_1)m_8(z_2)m_8(z_3)$ with m_8 in [16] satisfies conditions (20) and $\sum_\sigma |A^\sigma|^2 = 1$. The properties of M and A imply that

1. P satisfies the sub-QMF condition, since M is G -invariant and $0 \leq 1 - cM \leq 1$,
2. P satisfies sum rules of order at least 6,
3. the Taylor expansion of $1 - P^2$ at $z = \mathbf{1}$, in local coordinates (y_1, y_2, y_3) , has $2cM$ as its homogeneous part of lowest degree.

Therefore, $1 - P^2$ is not sos of trigonometric polynomials in $\mathbb{R}[T]$.

Next, we give an example of a trigonometric polynomial P satisfying $P(\mathbf{1}) = 1$ and sum rules of order at least 2, but its corresponding polynomial f is negative for some $\omega \in \mathbb{R}^3$.

Example 4.15. Consider

$$\begin{aligned}
P(z_1, z_2, z_3) = & 6z_1z_2z_3 \left(\frac{1+z_1}{2}\right)^2 \left(\frac{1+z_2}{2}\right)^2 \left(\frac{1+z_3}{2}\right)^2 \left(\frac{1+z_1z_2z_3}{2}\right)^2 - \\
& \frac{5}{4}z_1 \left(\frac{1+z_1}{2}\right) \left(\frac{1+z_2}{2}\right)^3 \left(\frac{1+z_3}{2}\right)^3 \left(\frac{1+z_1z_2z_3}{2}\right)^3 - \\
& \frac{5}{4}z_2 \left(\frac{1+z_1}{2}\right)^3 \left(\frac{1+z_2}{2}\right) \left(\frac{1+z_3}{2}\right)^3 \left(\frac{1+z_1z_2z_3}{2}\right)^3 - \\
& \frac{5}{4}z_3 \left(\frac{1+z_1}{2}\right)^3 \left(\frac{1+z_2}{2}\right)^3 \left(\frac{1+z_3}{2}\right) \left(\frac{1+z_1z_2z_3}{2}\right)^3 - \\
& \frac{5}{4}z_1z_2z_3 \left(\frac{1+z_1}{2}\right)^3 \left(\frac{1+z_2}{2}\right)^3 \left(\frac{1+z_3}{2}\right)^3 \left(\frac{1+z_1z_2z_3}{2}\right).
\end{aligned}$$

The associated refinable function is in $C(\mathbb{R}^3)$ as the corresponding subdivision scheme is uniformly convergent, but P does not satisfy the sub-QMF condition, as

$$1 - \sum_{\sigma \in G} |P^\sigma(e^{-i\omega})|^2 < 0 \quad \text{for } \omega = \left(\frac{\pi}{6}, 0, 0\right).$$

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